

# On quasiinvariants of $S_n$ of hook shape

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## Abstract

Chalykh, Veselov and Feigin introduced the notions of quasiinvariants for Coxeter groups, which is a generalization of invariants. In [2], Bandlow and Musiker showed that for the symmetric group  $S_n$  of order  $n$ , the space of quasiinvariants has a decomposition indexed by standard tableaux. They gave a description of basis for the components indexed by standard tableaux of shape  $(n-1, 1)$ . In this paper, we generalize their results to a description of basis for the components indexed by standard tableaux of arbitrary hook shape.

## 1 Introduction

In [3] and [5], Chalykh, Veselov and Feigin introduced the notions of *quasi invariants* for Coxeter groups, which is a generalization of invariants. For any Coxeter group, the quasiinvariants is defined by giving a multiplicity  $m$  which is a map from the conjugacy classes of the corresponding group to non-negative integers.

In the case of  $S_n$ , the multiplicity is a constant function. Take a non-negative integer  $m$ . A polynomial  $P \in \mathbb{Q}[x_1, x_2, \dots, x_n]$  is called a  $m$ -quasiinvariants if the difference

$$(1 - (i, j))P(x_1, \dots, x_n)$$

is divisible by  $(x_i - x_j)^{2m+1}$  for any transposition  $(i, j) \in S_n$ .

The notion of quasiinvariants was first introduced in the study of the quantum Calogero Moser system. In the case of  $S_n$ , this system is defined by

the following differential operator (the generalized Calogero-Moser Hamiltonian):

$$L_m = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - 2m \sum_{1 \leq i < j \leq n} \frac{1}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)$$

where  $m$  is a real number.

For a Coxeter group  $G$ , we denote by  $S^G$  the ideal generated by homogeneous invariant polynomials for  $G$  and by  $S_+^G$  the ideal generated by the homogeneous invariant polynomials of positive degree. For a generic multiplicity, there exist an isomorphism from  $S^G$  to the ring of  $G$ -invariant quantum integrals of the generalized Calogero-Moser Hamiltonian (sometimes called Harish-Chandra isomorphism). We denote by  $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$  the operators corresponding to fundamental invariant polynomials  $\sigma_1, \sigma_2, \dots, \sigma_n$ . The generalized Calogero-Moser Hamiltonian is a member of this ring (see for example [5], [6]).

In the case of non-negative integer multiplicities, Chalykh and Veselov showed that there is a homomorphism from the ring of quasiinvariants to the commutative ring of differential operators whose coefficients are rational functions (see e.g. [3]). It is shown that the restriction of such homomorphism onto  $S^G$  induces Harish-Chandra isomorphism. Therefore, in the case of non-negative integer multiplicities there are much more quantum integrals.

Let  $m$  be a non-negative multiplicity. In [5], Feigin and Veselov introduced the notions of  $m$ -harmonics which are defined as the solutions of the following system:

$$\begin{aligned} \mathcal{L}_1 \psi &= 0 \\ \mathcal{L}_2 \psi &= 0 \\ &\vdots \\ \mathcal{L}_n \psi &= 0. \end{aligned}$$

Feigin and Veselov also showed that the solutions of such system are polynomials. They also showed that the space of  $m$ -harmonic polynomials is a subspace of  $m$ -quasiinvariants of dimension  $|G|$ . In [7], Felder and Veselov gave a formula of the Hilbert series of the space of  $m$ -harmonic polynomials.

In [4], Etingof and Ginzburg proved the following: (i) quasiinvariants of  $G$  is a free module over  $S^G$  and the ring of quasiinvariants is Cohen-Macaulay and Gorenstein (ii) there is an isomorphism from the dual space of the quotient of quasiinvariants by  $S^G$  to the space of  $m$ -harmonic polynomials (iii) the

Hilbert series of the quotient of the quasiinvariants by  $S_+^G$  is equal to that of  $m$ -harmonic polynomials.

Let  $I_2(N)$  be the dihedral group of regular  $N$ -gon. In [5], Feigin and Veselov considered quasiinvariants for  $I_2(N)$  for a constant multiplicity. Since  $I_2(N)$  is of rank 2, quasiinvariants are expressed as essentially one variable. Feigin and Veselov gave generators over  $S^{I_2(N)}$  by a direct calculation. In [6], Feigin studied quasiinvariants for  $I_2(N)$  for any non-negative multiplicity. He gave a free basis of quasiinvariants over  $S^{I_2(N)}$  using the above mentioned results of Etingof and Ginzburg. An explicit description of basis of the quotient of quasiinvariants for  $S_3$  is contained in [5]. Another description is given in [1]. In [7], for  $S_n$  Felder and Veselov provide integral expressions for the lowest degree non-symmetric quasiinvariant polynomials (the degree  $nm+1$ ). However, for any integer  $n \geq 4$  a basis of the quotient of quasiinvariants for  $S_n$  is not known.

In this paper, we consider quasiinvariants for  $S_n$ . In this case,  $m$  is a non-negative integer. We denote these quasiinvariants by  $\mathbf{QI}_m$  and by  $\Lambda_n$  the space of symmetric polynomials. We define  $\mathbf{QI}_m^*$  as the quotient of  $\mathbf{QI}_m$  by the ideal generated by the homogeneous symmetric polynomials of positive degree.

In [2], Bandlow and Musiker showed that  $\mathbf{QI}_m$  has a decomposition indexed by standard tableaux. Each component has a  $\Lambda_n$  module structure. This decomposition can be extended to that of  $\mathbf{QI}_m^*$ . They constructed explicit basis of the submodules of  $\mathbf{QI}_m^*$  indexed by standard tableaux of shape  $(n-1, 1)$ .

In this paper, we extend the result in [2]. We construct basis of the submodules of  $\mathbf{QI}_m^*$  indexed by standard tableaux of shape  $(n-k+1, 1^{k-1})$  (a hook) (see Theorem.3.8). The elements of our basis are expressed as determinants of a matrix with entries similar to elements of basis introduced in [2]. We also show that our basis is a free basis of the submodule of  $\mathbf{QI}_m$  indexed by a hook  $(n-k+1, 1^{k-1})$  over  $\Lambda_n$  (Corollary.3.11).

We also show how the operator  $L_m$  acts on our basis. In [5], it is proved that the operator  $L_m$  preserves  $\mathbf{QI}_m$ . In [2], it is obtained explicit formulas of the action of  $L_m$  on their basis. We extend this formulas to that of our basis (Theorem.4.4).

## 2 Preliminaries

### 2.1 Symmetric group and Young diagram

We denote  $\mathbb{Q}[x_1, x_2, \dots, x_n]$  by  $K_n$  and the symmetric group on  $\{1, 2, \dots, n\}$  by  $S_n$ . For the finite set  $X$ , we denote the symmetric group on  $X$  by  $S_X$ .

The symmetric group  $S_n$  acts on  $K_n$  by

$$\sigma P(x_1, \dots, x_n) = P(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad \sigma \in S_n.$$

A polynomial  $P(x_1, x_2, \dots, x_n)$  is called a symmetric polynomial when for any  $\sigma \in S_n$   $P(x_1, x_2, \dots, x_n)$  satisfies

$$\sigma P(x_1, \dots, x_n) = P(x_1, \dots, x_n).$$

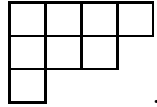
We denote by  $\Lambda_n$  the subspace spanned by symmetric polynomials and by  $\Lambda_n^d$  the subspace of  $\Lambda_n$  spanned by homogeneous polynomials of degree  $d$ . We set  $\Lambda_n^d = \{0\}$  if  $d < 0$ . The  $i$ -th elementary symmetric polynomial is denoted by  $e_i$ . For a partition  $\nu = (\nu_1, \nu_2, \dots)$ , we define  $e_\nu = \prod_i e_{\nu_i}$ . A basis of  $\Lambda_n$  is given by  $\{e_\nu\}$ .

The group ring  $S_n$  over  $\mathbb{Q}$  is denoted by  $\mathbb{Q}S_n$ . The action of  $S_n$  on  $K_n$  is naturally extended to that of  $\mathbb{Q}S_n$ . We define the elements of  $\mathbb{Q}S_n$ . For a subgroup  $H$  of  $S_n$ , we define  $[H], [H]'$  by

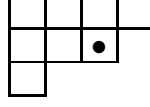
$$\begin{aligned} [H] &= \sum_{\sigma \in H} \sigma \\ [H]' &= \sum_{\sigma \in H} \text{sgn}(\sigma) \sigma. \end{aligned}$$

Let  $\lambda = (\lambda_1, \lambda_2, \dots)$  be a partition. When  $\lambda$  is a partition of a positive integer  $n$ , we denote this by  $\lambda \vdash n$ . We define  $l(\lambda) = \#\{i | \lambda_i \neq 0\}$  and  $|\lambda| = \sum_i \lambda_i$ . They are called the length and the size of  $\lambda$  respectively.

For a partition  $\lambda$ , the Young diagram of shape  $\lambda$  is a diagram such that its  $i$ -th row has  $\lambda_i$  boxes and it is arranged in left-justified rows. For example, the Young diagram of shape  $(4, 3, 1)$  is



We denote by  $(i, j)$  a box on the  $(i, j)$ -th position of the diagram. For instance, the box  $(2, 3)$  of the Young diagram of shape  $(4, 3, 1)$  is



We identify Young diagram of shape  $\lambda$  with a partition  $\lambda$ .

Let  $k, n$  be integers such that  $k \geq 2$  and  $n \geq k$ . We define  $\eta(n, k) = (n - k + 1, 1^{k-1})$ . We have  $l(\eta(n, k)) = k$  and  $|\eta(n, k)| = n$ . We call  $\eta(n, k)$  (also the tableau of shape  $\eta(n, k)$ ) the hook.

For  $\lambda \vdash n$ , we define the arm length  $a(i, j)$  for box  $(i, j) \in \lambda$  as

$$a(i, j) = \#\{(i, l) \mid j < l, (i, l) \in \lambda\}.$$

We also define the leg length  $l(i, j)$  for box  $(i, j)$  as

$$l(i, j) = \#\{(k, j) \mid i < k, (k, j) \in \lambda\}.$$

We define  $h(i, j) = a(i, j) + l(i, j) + 1$  called the hook length for box  $(i, j) \in \lambda$ .

A *tableau* of shape  $\lambda$  is a diagram filled in each box of  $\lambda$  with a positive integer. In this paper, we assume that entries of boxes are different each other. For a tableau  $D$ , we denote by  $D_{i,j}$  the entry in the box  $(i, j)$  of  $D$ . We define

$$\text{mem}(D) = \{D_{i,j} \mid (i, j) \in \lambda\}.$$

A tableau  $T$  is called a standard tableau if  $T$  satisfies  $\text{mem}(T) = \{1, 2, \dots, n\}$  and

$$T_{i,j} < T_{k,j}, \quad T_{i,j} < T_{i,l} \quad i < k, j < l.$$

We denote by  $ST(\lambda)$  the set of all standard tableaux of shape  $\lambda$  and by  $ST(n)$  the set of all standard tableaux with  $n$  boxes.

For a tableau  $D$  of shape  $\lambda$ , we define

$$C(D) = [\{\sigma \in S_{\text{mem}(D)} \mid \sigma \text{ preserves each column of } D\}]'$$

$$R(D) = [\{\sigma \in S_{\text{mem}(D)} \mid \sigma \text{ preserves each row of } D\}]$$

$$f_\lambda = |ST(\lambda)|$$

$$\gamma_D = \frac{f_\lambda C(D) R(D)}{n!}$$

$$V_D = \prod_{(i,j) \in C_D} (x_i - x_j)$$

where  $C_D = \{(i, j) \mid i < j \text{ and } i, j \text{ are the entries in same column of } D\}$ . The element  $\gamma_D \in \mathbb{Q}S_{\text{mem}(D)}$  satisfies  $\gamma_D^2 = \gamma_D$ .

**Definition 2.1** Let  $s_1, s_2, \dots, s_n$  be mutually distinct positive integers.

(1) We denote by  $D(s_1, s_2, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$  the tableau of shape  $\eta(n, k)$  such that the entries in the first column and in the first row are  $s_1, s_2, \dots, s_k$  and  $s_1, s_{k+1}, \dots, s_n$  in order, respectively.

(2) A tableau  $D(s_1, s_2, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$  is a standard tableau of shape  $\eta(n, k)$  if and only if the following holds:

$$\begin{aligned} s_1, s_2, \dots, s_n \text{ is a permutation of } 1, 2, \dots, n \\ s_1 = 1, s_2 \leq \dots \leq s_k, s_{k+1} \leq \dots \leq s_n. \end{aligned}$$

Then we simply write  $D(s_1, s_2, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$  as  $T(1, s_2, \dots, s_k)$ .

(3) Let  $i$  be a integer such that  $1 \leq i \leq k$  (resp.  $k+1 \leq i \leq n$ ). We set  $D = D(s_1, s_2, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$ . We define

$$\begin{aligned} D^{s_i} &= D(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_k; s_1, s_{k+1}, \dots, s_n) \\ (\text{resp. } D^{s_i} &= D(s_1, \dots, s_k; s_1, s_{k+1}, \dots, s_{i-1}, s_{i+1}, \dots, s_n)). \end{aligned}$$

For example, a standard tableau  $T(1, 3, 4) = D(1, 3, 4; 1, 2, 5, 6)$  of shape  $(4, 1, 1)$  is

|   |   |   |   |
|---|---|---|---|
| 1 | 2 | 5 | 6 |
| 3 |   |   |   |
| 4 |   |   |   |

The tableau  $T(1, 3, 4)^1$  is

|   |   |   |   |
|---|---|---|---|
| 3 | 2 | 5 | 6 |
| 4 |   |   |   |

and  $T(1, 3, 4)^2$  is

|   |   |   |
|---|---|---|
| 1 | 5 | 6 |
| 3 |   |   |
| 4 |   |   |

We have the following propositions.

**Proposition 2.2** ([2]) *For any  $f = \sum_{\sigma \in S_n} f_\sigma \sigma \in \mathbb{Q}S_n$ ,  $P \in \Lambda_n$  and  $Q \in K_n$ , we have  $f(PQ) = Pf(Q)$ .*

**Proposition 2.3** ([2]) *Let  $i_1, i_2, \dots, i_n$  be a permutation of  $1, 2, \dots, n$ . Then  $[S_n]$  and  $[S_n]'$  are expressed as follows:*

$$\begin{aligned} [S_n] &= (1 + (i_1, i_n) + \dots + (i_{n-1}, i_n)) \cdots (1 + (i_1, i_3) + (i_2, i_3)) (1 + (i_1, i_2)) \\ [S_n]' &= (1 - (i_1, i_n) - \dots - (i_{n-1}, i_n)) \cdots (1 - (i_1, i_3) - (i_2, i_3)) (1 - (i_1, i_2)). \end{aligned}$$

## 2.2 The quasiinvariants for $S_n$

We first recall results in [2].

**Lemma 2.4** ([2]) *The quasiinvariants  $\mathbf{QI}_m$  has following decomposition:*

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} \gamma_T(\mathbf{QI}_m).$$

$\gamma_T(\mathbf{QI}_m)$  has following description:

$$\gamma_T(\mathbf{QI}_m) = \gamma_T(K_n) \cap V_T^{2m+1} K_n. \quad (2.1)$$

For  $\lambda \vdash n$ , the vector space  $\bigoplus_{T \in ST(\lambda)} \gamma_T(\mathbf{QI}_m)$  is called the  $\lambda$ -isotypic component of  $\mathbf{QI}_m$ .

Let  $K$  be a polynomial ring. We denote by  $K[i]$  the subspace spanned by homogeneous polynomials of degree  $i$  in  $K$ . The Hilbert series of  $K$  is defined as a formal power series  $\sum_{i=0}^{\infty} \dim(K[i])t^i$ . We denote it by  $H(K, t)$ .

We denote  $\mathbf{QI}_m / \langle e_1, \dots, e_n \rangle$  by  $\mathbf{QI}_m^*$ . For  $[f] \in \mathbf{QI}_m^*$ , we define the degree of  $[f]$  as the minimal degree in the class  $[f]$ . In [4] and [7], the Hilbert series of  $\mathbf{QI}_m^*$  is given by as follows:

**Theorem 2.5** ([4], [7])

$$H(\mathbf{QI}_m^*, t) = n! t^{mn(n-1)/2} \sum_{\lambda \vdash n} \prod_{(i,j) \in \lambda} \prod_{k=1}^n t^{w(i,j;m)} \frac{1 - t^k}{h(i,j)(1 - t^{h(i,j)})} \quad (2.2)$$

where we set  $w(i, j; m) = m(l(i, j) - a(i, j)) + l(i, j)$ .

In particular, for  $T \in ST(\lambda)$  the Hilbert series of  $\gamma_T(\mathbf{QI}_m^*)$  is given as follows:

$$H(\gamma_T(\mathbf{QI}_m^*); t) = t^{mn(n-1)/2} \prod_{(i,j) \in \lambda} \prod_{k=1}^n t^{w(i,j;m)} \frac{1-t^k}{1-t^{h(i,j)}}. \quad (2.3)$$

Let  $s_1, s_2, \dots, s_n$  be mutually distinct positive integers. We set  $D = D(s_1, s_2; s_1, s_3, \dots, s_n)$ . We define the following polynomial in  $\mathbb{Q}[x_{s_1}, x_{s_2}, \dots, x_{s_n}]$ :

$$Q_D^{l;m} = \int_{x_{s_1}}^{x_{s_2}} t^l \prod_{i=1}^n (t - x_{s_i})^m dt. \quad (2.4)$$

Recall that we define  $\eta(n, k) = (n - k + 1, 1^{k-1})$ . In [2], J.Bandlow and G.Musiker found an explicit basis of  $\gamma_T(\mathbf{QI}_m^*)$  when  $T \in ST(\eta(n, 2))$ .

**Theorem 2.6 ([2])** *Let  $T \in ST(\eta(n, 2))$ .  $\{Q_T^{0;m}, Q_T^{1;m}, \dots, Q_T^{n-2;m}\}$  is a basis of  $\gamma_T(\mathbf{QI}_m^*)$ .*

**Remark 2.7** In [2], it is shown that  $Q_T^{l;m}$  is divisible by  $V_T = (x_1 - x_j)^{2m+1}$ . We can similarly show that  $Q_D^{l;m}$  is divisible by  $V_D = (x_{s_1} - x_{s_2})^{2m+1}$ .

Let  $f \in \mathbb{Q}[x_{s_1}, x_{s_2}, \dots, x_{s_n}]$ . We denote by  $\deg_{x_{s_i}}(f)$  the degree of  $f$  as the polynomial in  $x_{s_i}$ . For a homogeneous polynomial  $g$ , we define  $\deg(g)$  as the degree of  $g$ .

The polynomials  $Q_D^{l;m}$  have the following properties, which we will use to show Proposition.3.3.

**Proposition 2.8** *Let  $s_1, s_2, \dots, s_n$  be mutually distinct positive integers. Let  $l$  be a non-negative integer and take a tableau  $D = D(s_1, s_2; s_1, s_3, \dots, s_n)$  of shape  $\eta(n, 2)$ .*

*$Q_D^{l;m}$  is a homogeneous polynomial of degree  $nm + l + 1$  and satisfies following properties.*

- (1)  $Q_D^{l;m}$  is symmetric in  $x_{s_3}, \dots, x_{s_n}$  and anti-symmetric in  $x_{s_1}, x_{s_2}$ .
- (2) We have  $\deg_{x_{s_1}}(Q_D^{l;m}) = nm + l + 1$ . The leading coefficient of  $Q_D^{l;m}$  in  $x_{s_1}$  is  $\frac{(-1)^{m+1}m!}{\prod_{s=0}^m (mn + l + 1 - s)}$ .
- (3) Let  $i \in \{1, \dots, n\} \setminus \{1, 2\}$ . We have  $\deg_{x_{s_i}}(Q_D^{l;m}) = m$ . The leading coefficient of  $Q_D^{l;m}$  in  $x_{s_i}$  is equal to  $(-1)^m Q_D^{l;m}$ .



Proof We show the case  $D = T(1, 2)$  since the proofs of other cases are similar. We set  $T = T(1, 2)$ .

- (1) It follows from the fact that  $t^l \prod_{i=1}^n (t - x_i)^m$  is symmetric in  $x_1, x_2, \dots, x_n$ .  
(2) We show this statement by induction on  $m$ .

When  $m = 0$ ,  $Q_T^{l;0}$  is  $\frac{1}{l+1}(x_j^{l+1} - x_1^{l+1})$ , the statement holds.

When  $m \geq 1$ , assume that the statement holds for all numbers less than  $m$ . In [2], the polynomial  $Q_T^{l;m}$  is expressed as:

$$Q_T^{l;m} = \sum_{i=0}^n (-1)^i e_i Q_T^{n+l-i;m-1}. \quad (2.5)$$

By induction assumption on  $m$ , we have  $\deg_{x_{s_1}}(Q_T^{n+l-i;m-1}) = nm + l - i + 1$ . From (2.5), we have  $\deg_{x_1}(Q_T^{l;m}) = nm + l + 1$  and the term with the highest degree are in  $e_0 Q_T^{n+l;m-1} - e_1 Q_T^{n+l-1;m-1}$ . The leading coefficient of  $Q_T^{l;m}$  in  $x_1$  is

$$\begin{aligned} & \frac{(-1)^m(m-1)!}{\prod_{s=0}^{m-1}(mn+l+1-s)} - \frac{(-1)^m(m-1)!}{\prod_{s=0}^{m-1}(mn+l-s)} \\ &= \frac{(-1)^{m+1}m!}{\prod_{s=0}^m(mn+l+1-s)}. \end{aligned}$$

- (3) Expanding  $(t - x_i)^m$  in  $Q_T^{l;m}$ , we have

$$Q_T^{l;m} = \sum_{s=0}^m (-1)^s \binom{m}{s} Q_{T^i}^{l;m} x_i^s.$$

Thus propositions is proved.  $\square$

As a corollary of this proposition, we have  $Q_D^{l;m} \neq 0$  when  $D$  is a tableau of shape  $\eta(n, 2)$ .

### 3 A basis for the isotypic component of shape $(n - k + 1, 1^{k-1})$

We give a basis for the  $\eta(n, k)$ -isotypic component. Let  $s_1, s_2, \dots, s_n$  be mutually distinct positive integers. Throughout this section, we set  $D = D(s_1, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$  and  $T = T(1, 2, \dots, k)$ .

Definition 3.1 (1) Let  $p$  be a non-negative integer. For  $i, j$  such that  $1 \leq i < j \leq k$ , we define a polynomial  $R_{D;s_i,s_j}^{p;m}$  in  $\mathbb{Q}[x_{s_1}, x_{s_2}, \dots, x_{s_n}]$  as

$$R_{D;s_i,s_j}^{p;m} = \int_{x_{s_i}}^{x_{s_j}} t^p \prod_{l=1}^n (t - x_{s_l})^m dt. \quad (3.1)$$

(2) Let  $k$  be an integer such that  $k \geq 2$ . Take a partition  $\mu = (\mu_1, \mu_2, \dots, \mu_{k-1})$  such that  $\mu_1 > \mu_2 > \dots > \mu_{k-1} \geq 0$ . We define a polynomial  $Q_D^{\mu;m}$  in  $\mathbb{Q}[x_{s_1}, x_{s_2}, \dots, x_{s_n}]$  as follows:

$$Q_D^{\mu;m} = \begin{vmatrix} R_{D;s_1,s_2}^{\mu_1;m} & R_{D;s_1,s_2}^{\mu_2;m} & \cdots & R_{D;s_1,s_2}^{\mu_{k-1};m} \\ R_{D;s_2,s_3}^{\mu_1;m} & R_{D;s_2,s_3}^{\mu_2;m} & \cdots & R_{D;s_2,s_3}^{\mu_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{D;s_{k-1},s_k}^{\mu_1;m} & R_{D;s_{k-1},s_k}^{\mu_2;m} & \cdots & R_{D;s_{k-1},s_k}^{\mu_{k-1};m} \end{vmatrix}. \quad (3.2)$$

We denote the empty sequence by  $\emptyset$ . When  $k = 1$ ,  $\mu$  is the empty sequence  $\emptyset$ . We set  $Q_D^{\emptyset;m} = 1$ . We simply write  $Q_D^m$  as  $Q_D^{\emptyset;m}$ .

Remark 3.2 Setting  $D' = D(s_1, s_2; s_1, s_3, \dots, s_n)$ , we have  $R_{D;s_1,s_2}^{p;m} = Q_{D'}^{p;m}$ .

The polynomials  $Q_D^{\mu;m}$  have the following properties, which we will use to show our main results.

**Proposition 3.3** *Let  $s_1, s_2, \dots, s_n$  be mutually distinct positive integers. We set  $D = D(s_1, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$ . Let  $\mu = (\mu_1, \mu_2, \dots, \mu_{k-1})$  be a partition such that  $\mu_1 > \mu_2 > \dots > \mu_{k-1} \geq 0$ .*

*Then,  $Q_D^{\mu;m}$  satisfies the following.*

- (1)  $Q_D^{\mu;m}$  is symmetric in  $x_{s_{k+1}}, x_{s_{k+2}}, \dots, x_{s_n}$  and anti-symmetric in  $x_{s_1}, x_{s_2}, \dots, x_{s_k}$ . In particular,  $Q_D^{\mu;m}$  is divisible by  $V_D^{2m+1}$ .
- (2) We have  $\deg_{x_{s_1}}(Q_D^{\mu;m}) = (n + k - 2)m + \mu_1 + 1$ . The leading coefficient of  $Q_D^{\mu;m}$  in  $x_{s_1}$  is

$$\frac{(-1)^{(k-1)m+1} m!}{\prod_{s=0}^m (mn + \mu_1 + 1 - s)} Q_{D^{s_1}}^{(\mu_2, \dots, \mu_{k-1});m}.$$

*In particular, we have  $\deg(Q_D^{\mu;m}) = (k - 1)nm + |\mu| + k - 1$ .*

- (3) We have  $\deg_{x_{k+1}}(Q_D^{\mu;m}) = (k - 1)m$ . The leading coefficient of  $Q_D^{\mu;m}$  in  $x_{k+1}$  is  $(-1)^{(k-1)m} Q_{D^{s_{k+1}}}^{\mu;m}$ .
- (4) Polynomial  $Q_D^{\mu;m}$  is invariant under  $\gamma_D$ .

Proof We show the case  $D = T$ . The proofs of other cases are similar.

(1) From Prop.2.8(1), it follows that  $Q_T^{\mu;m}$  is symmetric in  $x_{k+1}, x_{k+2}, \dots, x_n$ .

Adding the first row to the second row, we get

$$Q_T^{\mu;m} = \begin{vmatrix} R_{T;1,2}^{\mu_1;m} & R_{T;1,2}^{\mu_2;m} & \cdots & R_{T;1,2}^{\mu_{k-1};m} \\ R_{T;1,3}^{\mu_1;m} & R_{T;1,3}^{\mu_2;m} & \cdots & R_{T;1,3}^{\mu_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu_1;m} & R_{T;k-1,k}^{\mu_2;m} & \cdots & R_{T;k-1,k}^{\mu_{k-1};m} \end{vmatrix}.$$

Repeating this process, we get

$$Q_T^{\mu;m} = \begin{vmatrix} R_{T;1,2}^{\mu_1;m} & R_{T;1,2}^{\mu_2;m} & \cdots & R_{T;1,2}^{\mu_{k-1};m} \\ R_{T;1,3}^{\mu_1;m} & R_{T;1,3}^{\mu_2;m} & \cdots & R_{T;1,3}^{\mu_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;1,k}^{\mu_1;m} & R_{T;1,k}^{\mu_2;m} & \cdots & R_{T;1,k}^{\mu_{k-1};m} \end{vmatrix}. \quad (3.3)$$

Thus,  $Q_T^{\mu;m}$  is anti-symmetric in  $x_2, \dots, x_k$ . We can show that  $Q_T^{\mu;m}$  is anti-symmetric in  $x_1, x_3, \dots, x_k$  and  $x_1, x_2, x_4, \dots, x_k$  in the similar way. Thus the first statement holds.

From Remark.2.7 and (3.3),  $Q_T^{\mu;m}$  is divisible by  $\prod_{s=2}^n (x_1 - x_s)^{2m+1}$ . Using this proposition (1) we see  $Q_T^{\mu;m}$  is also divisible by  $V_T^{2m+1}$ .

(2) We see  $Q_T^{\mu;m}$  as a polynomial in  $x_1$ . From Prop.2.8 (2),(3), the term of the degree of  $Q_T^{\mu;m}$  is in  $R_{T;1,2}^{\mu_1;m} R_{T;2,3}^{\mu_2;m} \cdots R_{T;k-1,k}^{\mu_k;m}$ . We use Prop.2.8 (2),(3) again, therefore the statement holds.

(3) From Prop.2.8 (3), the leading coefficient of  $x_{k+1}$  is

$$\begin{vmatrix} (-1)^m R_{T^{k+1};1,2}^{\mu_1;m} & (-1)^m R_{T^{k+1};1,2}^{\mu_2;m} & \cdots & (-1)^m R_{T^{k+1};1,2}^{\mu_k;m} \\ (-1)^m R_{T^{k+1};2,3}^{\mu_1;m} & (-1)^m R_{T^{k+1};2,3}^{\mu_2;m} & \cdots & (-1)^m R_{T^{k+1};2,3}^{\mu_k;m} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^m R_{T^{k+1};k-1,k}^{\mu_1;m} & (-1)^m R_{T^{k+1};k-1,k}^{\mu_2;m} & \cdots & (-1)^m R_{T^{k+1};k-1,k}^{\mu_k;m} \end{vmatrix}. \quad (3.4)$$

The polynomial (3.4) is equal to  $(-1)^{(k-1)m} Q_{T^{k+1}}^{\mu;m}$ .

(4) To prove (4), we define the following notion.

For positive integers  $i, j$  such that  $i \neq j$ , we define a tableau  $(i, j)D$  as follows. When  $i, j \notin \text{mem}(D)$ , we define  $(i, j)D = D$ . When  $i \in \text{mem}(D)$  and  $j \notin \text{mem}(D)$ ,  $(i, j)D$  is a tableau obtained by replacing the entry  $i$  in  $D$  with  $j$ . When  $i, j \in \text{mem}(D)$ ,  $(i, j)D$  is a tableau obtained by interchanging the entry  $i$  and  $j$  in  $D$ .

Using Prop.2.3,  $\gamma_T$  is

$$\frac{1}{n(n-k)!(k-1)!} \left\{ 1 - \sum_{s=2}^k (1, s) \right\} [S_{\{2,3,\dots,k\}}]' \left\{ 1 + \sum_{s=k+1}^n (1, s) \right\} [S_{\{k+1,\dots,n\}}].$$

Therefore from (1),

$$\gamma_T(Q_T^{\mu;m}) = \frac{1}{n} \left\{ k Q_T^{\mu;m} + \sum_{s=k+1}^n \{1 - (1, 2) - \dots - (1, k)\} Q_{(1,s)T}^{\mu;m} \right\}.$$

We consider the sum  $\sum_{s=k+1}^n \{1 - (1, 2) - \dots - (1, k)\} Q_{(1,s)T}^{\mu;m}$ .

$$\begin{aligned} & \sum_{s=k+1}^n \{1 - (1, 2) - (1, 3) - \dots - (1, k)\} Q_{(1,s)T}^{\mu;m} \\ &= \sum_{s=k+1}^n \{Q_{(1,s)T}^{\mu;m} + Q_{(2,s)T}^{\mu;m} + Q_{(3,s)T}^{\mu;m} + \dots + Q_{(k,s)T}^{\mu;m}\}. \end{aligned}$$

Consider the sum  $Q_{(1,s)T}^{\mu;m} + Q_{(2,s)T}^{\mu;m}$ . By definition, we have

$$\begin{aligned} & Q_{(1,s)T}^{\mu;m} + Q_{(2,s)T}^{\mu;m} \\ &= \begin{vmatrix} R_{T;s,2}^{\mu_1;m} & R_{T;s,2}^{\mu_2;m} & \cdots & R_{T;s,2}^{\mu_{k-1};m} \\ R_{T;2,3}^{\mu_1;m} & R_{T;2,3}^{\mu_2;m} & \cdots & R_{T;2,3}^{\mu_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu_1;m} & R_{T;k-1,k}^{\mu_2;m} & \cdots & R_{T;k-1,k}^{\mu_{k-1};m} \end{vmatrix} + \begin{vmatrix} R_{T;1,s}^{\mu_1;m} & R_{T;1,s}^{\mu_2;m} & \cdots & R_{T;1,s}^{\mu_{k-1};m} \\ R_{T;s,3}^{\mu_1;m} & R_{T;s,3}^{\mu_2;m} & \cdots & R_{T;s,3}^{\mu_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu_1;m} & R_{T;k-1,k}^{\mu_2;m} & \cdots & R_{T;k-1,k}^{\mu_{k-1};m} \end{vmatrix}. \end{aligned}$$

Adding the first row to the second row in the second determinant, we get

$$\begin{aligned} & Q_{(1,s)T}^{\mu;m} + Q_{(2,s)T}^{\mu;m} \\ &= \begin{vmatrix} R_{T;s,2}^{\mu_1;m} & R_{T;s,2}^{\mu_2;m} & \cdots & R_{T;s,2}^{\mu_{k-1};m} \\ R_{T;s,3}^{\mu_1;m} & R_{T;s,3}^{\mu_2;m} & \cdots & R_{T;s,3}^{\mu_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu_1;m} & R_{T;k-1,k}^{\mu_2;m} & \cdots & R_{T;k-1,k}^{\mu_{k-1};m} \end{vmatrix} + \begin{vmatrix} R_{T;1,s}^{\mu_1;m} & R_{T;1,s}^{\mu_2;m} & \cdots & R_{T;1,s}^{\mu_{k-1};m} \\ R_{T;s,3}^{\mu_1;m} & R_{T;s,3}^{\mu_2;m} & \cdots & R_{T;s,3}^{\mu_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu_1;m} & R_{T;k-1,k}^{\mu_2;m} & \cdots & R_{T;k-1,k}^{\mu_{k-1};m} \end{vmatrix}. \end{aligned}$$

Adding the two term, we have

$$\begin{aligned}
& Q_{(1,s)T}^{\mu;m} + Q_{(2,s)T}^{\mu;m} \\
&= \begin{vmatrix} R_{T;1,2}^{\mu_1;m} & R_{T;1,2}^{\mu_2;m} & \cdots & R_{T;1,2}^{\mu_{k-1};m} \\ R_{T;s,3}^{\mu_1;m} & R_{T;s,3}^{\mu_2;m} & \cdots & R_{T;s,3}^{\mu_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\mu_1;m} & R_{T;k-1,k}^{\mu_2;m} & \cdots & R_{T;k-1,k}^{\mu_{k-1};m} \end{vmatrix}.
\end{aligned}$$

Repeating this process, we get

$$\{1 - (1, 2) - (1, 3) - \cdots - (1, k)\} Q_{(1,s)T}^{\mu;m} = Q_T^{\mu;m}.$$

Thus, proposition is proved.  $\square$

As a corollary of this proposition, we have  $Q_T^{\mu;m} \in \gamma_T(\mathbf{QI_m})$  where  $T \in ST(\eta(n, k))$ .

We introduce the following notions.

**Definition 3.4** Let  $s, t, u$  be non-negative integers. When  $u \geq 1$ , we set the subsets of partitions  $P(s; t; u)$ ,  $P(t; u)$  and  $Q(s; t; u)$  as:

$$\begin{aligned}
P(s; t; u) &= \{\lambda \mid |\lambda| = s, t \geq \lambda_1 > \lambda_2 > \cdots > \lambda_u \geq 0\} \\
Q(s; t; u) &= P(s; t; u) \setminus P(s; t-1; u) \\
P(t; u) &= \cup_{s \geq 0} P(s; t; u).
\end{aligned}$$

When  $u = 0$ , we set

$$\begin{aligned}
P(0; t; 0) &= \{\emptyset\} \\
P(t; 0) &= \{\emptyset\}.
\end{aligned}$$

Let  $l$  be a positive integer. We set  $P(l; t; 0)$  as empty set.

We define  $p(s; t; u) = \sharp P(s; t; u)$  and  $q(s; t; u) = \sharp Q(s; t; u)$ .

**Remark 3.5** Setting  $\mu \in P(n-2; k-1)$  (resp.  $\mu \in \cup_{s \geq 0} Q(s; n-2; k-1)$ ), we have

$$\begin{aligned}
\frac{(k-1)(k-2)}{2} \leq |\mu| &\leq (k-1)(n-k) + \frac{(k-1)(k-2)}{2} \\
\left( \text{resp. } n-2 + \frac{(k-2)(k-3)}{2} \leq |\mu| \right. &\leq (k-1)(n-k) + \frac{(k-1)(k-2)}{2} \left. \right).
\end{aligned}$$

We have the following proposition.

**Proposition 3.6** *Let  $k$  be an integer such that  $k \geq 2$ .*

(1) *Let  $l$  be a integer such that  $0 \leq l \leq n - k - 1$ . Then, we have*

$$p\left(l + \frac{(k-1)(k-2)}{2}; n-3; k-1\right) = p\left(l + \frac{(k-1)(k-2)}{2}; n-2; k-1\right).$$

(2) *Let  $l$  be a integer such that  $l \geq n - k$ . Then, we have*

$$\begin{aligned} & p\left(l + \frac{(k-1)(k-2)}{2}; n-2; k-1\right) \\ &= p\left(l + \frac{(k-1)(k-2)}{2}; n-3; k-1\right) \\ &+ p\left(l + k - n + \frac{(k-2)(k-3)}{2}; n-3; k-2\right). \end{aligned}$$

(3) *Let  $l \in \{0, 1, \dots, k-2\}$ . Then, we have*

$$\begin{aligned} & p\left((k-1)(n-k) + \frac{(k-1)(k-2)}{2} - l; n-2; k-1\right) \\ &= p\left((k-2)(n-k) + \frac{(k-2)(k-3)}{2} - l; n-3; k-2\right). \end{aligned}$$

Proof (1) By definition, we have

$$\begin{aligned} & q\left(l + \frac{(k-1)(k-2)}{2}; n-2; k-1\right) \\ &= p\left(l + \frac{(k-1)(k-2)}{2}; n-2; k-1\right) - p\left(l + \frac{(k-1)(k-2)}{2}; n-3; k-1\right). \end{aligned}$$

Therefore we show  $q\left(l + \frac{(k-1)(k-2)}{2}; n-2; k-1\right) = 0$ .

We have  $l + \frac{(k-1)(k-2)}{2} \leq n - k - 1 + \frac{(k-1)(k-2)}{2} < n - 2 + \frac{(k-2)(k-3)}{2}$ . From Remark.3.5, proposition follows.

(2) To prove (2), we show

$$\begin{aligned} & q\left(l + \frac{(k-1)(k-2)}{2}; n-2; k-1\right) \\ &= p\left(l + k - n + \frac{(k-2)(k-3)}{2}; n-3; k-2\right). \end{aligned}$$

By definition, we have

$$q\left(l + \frac{(k-1)(k-2)}{2}; n-2; k-1\right) = \sum_{s=0}^{n-3} q\left(l + \frac{(k-1)(k-2)}{2} - n+2; s; k-2\right).$$

We have  $l + \frac{(k-1)(k-2)}{2} - n+2 = l + k - n + \frac{(k-2)(k-3)}{2}$ , therefore we get

$$\begin{aligned} & q\left(l + \frac{(k-1)(k-2)}{2}; n-2; k-1\right) \\ &= \sum_{s=0}^{n-3} q\left(l + k - n + \frac{(k-2)(k-3)}{2}; s; k-2\right). \end{aligned}$$

By definition, we obtain

$$\begin{aligned} & \sum_{s=0}^{n-3} q\left(l + k - n + \frac{(k-2)(k-3)}{2}; s; k-2\right) \\ &= p\left(l + k - n + \frac{(k-2)(k-3)}{2}; n-3; k-2\right). \end{aligned}$$

(3) By definition, we have

$$\begin{aligned} & p\left((k-1)(n-k) + \frac{(k-1)(k-2)}{2} - l; n-2; k-1\right) \\ &= \sum_{s=0}^{n-2} q\left((k-1)(n-k) + \frac{(k-1)(k-2)}{2} - l; s; k-1\right). \end{aligned}$$

From Remark.3.5, we have  $q\left((k-1)(n-k) + \frac{(k-1)(k-2)}{2} - l; s; k-1\right) = 0$  when  $s \leq n-3$ . Therefore we obtain

$$\begin{aligned} & p\left((k-1)(n-k) + \frac{(k-1)(k-2)}{2} - l; n-2; k-1\right) \\ &= q\left((k-1)(n-k) + \frac{(k-1)(k-2)}{2} - l; n-2; k-1\right). \end{aligned}$$

From (2), we have

$$\begin{aligned} & q\left((k-1)(n-k) + \frac{(k-1)(k-2)}{2} - l; n-2; k-1\right) \\ &= p\left((k-1)(n-k) + \frac{(k-2)(k-3)}{2} - l + k - n; n-3; k-2\right) \\ &= p\left((k-2)(n-k) + \frac{(k-2)(k-3)}{2} - l; n-3; k-2\right). \quad \square \end{aligned}$$

We next consider the Hilbert series of  $\gamma_T(\mathbf{QI}_m^*)$ . To simplified notions, we define  $p_{s,n-2,k-1} = p\left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right)$ .

Proposition.3.6 is written as:

- (1)  $p_{l,n-3,k-1} = p_{l,n-2,k-1}$
- (2)  $p_{l,n-2,k-1} = p_{l,n-3,k-1} + p_{l+k-n,n-3,k-2}$
- (3)  $p_{(k-1)(n-k)-l,n-2,k-1} = p_{(k-2)(n-k)-l,n-3,k-2}$ .

**Lemma 3.7** *We have*

$$H(\gamma_T(\mathbf{QI}_m^*); t) = t^{(k-1)nm + \frac{k(k-1)}{2}} \sum_{s=0}^{(k-1)(n-k)} p_{s,n-2,k-1} t^s. \quad (3.5)$$

Proof From (2.3),  $H(\gamma_T(\mathbf{QI}_m^*); t)$  is equal to

$$t^{mn(n-1)/2} \prod_{(i,j) \in \lambda} \prod_{l=1}^n t^{m(l(i,j)-a(i,j))+l(i,j)} \frac{1-t^l}{1-t^{h(i,j)}}.$$

For  $2 \leq i \leq n-k+1$  and  $2 \leq j \leq k$ , we have

$$\begin{aligned} a(1,1) &= n-k, \quad l(1,1) = k-1, \quad h(1,1) = n \\ a(1,i) &= n-k+1-i, \quad l(1,i) = 0, \quad h(1,i) = n-k+2-i \\ a(j,1) &= 0, \quad l(j,1) = k-j, \quad h(j,1) = k-j+1. \end{aligned}$$

Thus we have

$$H(\gamma_T(\mathbf{QI}_m^*); t) = t^{(k-1)nm + \frac{k(k-1)}{2}} \prod_{s=1}^{k-1} \frac{(1-t^{n-s})}{(1-t^s)}.$$

Therefore, we must show

$$\prod_{s=1}^{k-1} \frac{(1-t^{n-s})}{(1-t^s)} = \sum_{s=0}^{(k-1)(n-k)} p_{s,n-2,k-1} t^s. \quad (3.6)$$

We show this by induction on  $n$ .

If  $n = k$ , then both of l.h.s and r.h.s are both equal to 1.



When  $n \geq k + 1$ , we assume that (3.6) holds with all numbers less than  $n$ . We have the following identity:

$$\prod_{s=1}^{k-1} \frac{(1 - t^{n-s})}{(1 - t^s)} = \prod_{s=1}^{k-1} \frac{(1 - t^{n-s-1})}{(1 - t^s)} + t^{n-k} \prod_{s=1}^{k-2} \frac{(1 - t^{n-s-1})}{(1 - t^s)}.$$

By induction assumption, we obtain

$$\begin{aligned} & \prod_{s=1}^{k-1} \frac{(1 - t^{n-s-1})}{(1 - t^s)} + t^{n-k} \prod_{s=1}^{k-2} \frac{(1 - t^{n-s-1})}{(1 - t^s)} \\ &= \sum_{s=0}^{(k-1)(n-k-1)} p_{s,n-3,k-1} t^s + t^{n-k} \sum_{s=0}^{(k-2)(n-k)} p_{s,n-3,k-2} t^s. \end{aligned}$$

We can rewrite this as

$$\begin{aligned} & \prod_{s=1}^{k-1} \frac{(1 - t^{n-s})}{(1 - t^s)} \\ &= \sum_{s=(k-1)(n-k)-k+2}^{(k-1)(n-k)} p_{s-n+k,n-3,k-2} t^s \\ &+ \sum_{s=n-k}^{(k-1)(n-k-1)} (p_{s-n+k,n-3,k-2} + p_{s,n-3,k-1}) t^s \\ &+ \sum_{s=0}^{n-k-1} p_{s,n-3,k-1} t^s. \end{aligned}$$

Using Proposition.3.6 (2), we have

$$\begin{aligned} & \sum_{s=n-k}^{(k-1)(n-k-1)} (p_{s-n+k,n-3,k-2} + p_{s,n-3,k-1}) t^s \\ &= \sum_{s=n-k}^{(k-1)(n-k-1)} p_{s,n-2,k-1} t^s. \end{aligned}$$

From Proposition.3.6 (1) and (3), lemma holds.  $\square$

We state the main theorem in this paper.

**Theorem 3.8**  $\{Q_T^{\mu;m}\}_{\mu \in P(n-2;k-1)}$  is a basis of  $\gamma_T(\mathbf{QI}_m^*)$ .

To simplified notions, we set

$$\begin{aligned} P_{s,n-2,k-1} &= P\left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right) \\ P_{n-2,k-1} &= P(n-2; k-1) \\ Q_{s,n-2,k-1} &= Q\left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right). \end{aligned}$$

We define following notions.

Let  $X = \{s_1, s_2, \dots, s_n\}$  be the set of  $n$  positive integers. We recall that  $S_X$  is the symmetric group on  $X$  and  $S_X$  acts on  $\mathbb{Q}[x_{s_1}, x_{s_2}, \dots, x_{s_n}]$  on the left.

We define  $\Lambda_X$  as the subspace of  $\mathbb{Q}[x_{s_1}, x_{s_2}, \dots, x_{s_n}]$  spanned by all polynomials which is invariant under  $S_X$ . We define  $\Lambda_X^d$  as the subspace of  $\Lambda_X$  spanned by homogeneous polynomials of degree  $d$ . We define  $\Lambda_X^d = \{0\}$  if  $d < 0$ .

Theorem.3.8 follows from the following proposition.

**Proposition 3.9** Let  $D$  be a tableau of shape  $\eta(n, k)$ . If

$$\sum_{\mu \in P(n-2;k-1)} f_\mu Q_D^{\mu;m} = 0 \tag{3.7}$$

where  $f_\mu \in \Lambda_{\text{mem}(D)}$ , then all  $f_\mu$  is equal to 0.

*Proof* We show this lemma by induction on the size of tableaux.

In the case  $k = 1$ , (3.9) is  $f Q_D^m = 0$  where  $f \in \Lambda_{\text{mem}(D)}$ . Therefore proposition holds when  $k = 1$ . We assume that  $k \geq 2$ .

We recall that  $n \geq k$ . When  $n = 2$ , we have  $k = 2$ . Then l.h.s of (3.7) is equal to  $f_0 Q_D^{0;m}$ . Therefore lemma holds when  $n = 2$ .

Assume that (3.7) holds when size is less than  $n$  for  $n \geq 3$ . We show the case  $D = T$  since the proofs of other cases are similar.

We recall that  $\Lambda_n$  is a graded ring. Therefore we can decompose

$$f_\mu = \sum_{l \geq 0} f_{\mu,l}$$

where  $f_{\mu,l} \in \Lambda_n^l$ . Thus, (3.7) is written as

$$\sum_{\mu \in P(n-2;k-1)} \sum_{l \geq 0} f_{\mu,l} Q_T^{\mu;m} = 0 \quad (3.8)$$

where  $f_{\mu,l} \in \Lambda_n^l$ . We have  $\deg(Q_T^{\mu;m}) = (k-1)nm + |\mu| + k - 1$ , therefore we obtain  $\deg(f_{\mu,l} Q_T^{\mu;m}) = (k-1)nm + |\mu| + d + k - 1$ .

Thus, (3.8) is written as

$$\sum_{d \geq 0} \sum_{\mu \in P(n-2;k-1)} f_{\mu,d-(k-1)nm-|\mu|-k+1} Q_T^{\mu;m} = 0. \quad (3.9)$$

Hence, for any  $d$  we obtain

$$\sum_{\mu \in P(n-2;k-1)} f_{\mu,d-(k-1)nm-|\mu|-k+1} Q_T^{\mu;m} = 0. \quad (3.10)$$

Fix  $d$ . Recall that the set  $P_{s,n-2,k-1}$  is not empty set if  $0 \leq s \leq (k-1)(n-k)$ . Let  $s$  be an integer such that  $0 \leq s \leq (k-1)(n-k)$  and take  $\mu \in P_{s,n-2,k-1}$ . Then we have  $\deg(Q_T^{\mu;m}) = (k-1)nm + \frac{k(k-1)}{2} + s$ . We set  $d' = d - (k-1)nm - \frac{k(k-1)}{2}$ . We express  $f_{\mu,d'-s}$  as

$$\sum_{r=0}^{d'-s} \sum_{\substack{|\nu|=d'-s \\ l(\nu)=r}} a_{r,\nu}^{\mu} e_{\nu}.$$

We recall that

$$\begin{aligned} P_{s,n-2,k-1} &= P\left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right) \\ P_{n-2,k-1} &= P(n-2; k-1) \\ Q_{s,n-2,k-1} &= Q\left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right). \end{aligned}$$

Therefore (3.10) is written as

$$\sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_{s,n-2,k-1}} \sum_{r=0}^{d'-s} \sum_{\substack{|\nu|=d'-s \\ l(\nu)=r}} a_{r,\nu}^{\mu} Q_T^{\mu;m} = 0. \quad (3.11)$$

We show  $a_{r,\nu}^\mu = 0$  for  $r \geq 0$ . We show this by induction on  $r$ . To prove this, we consider the highest degree terms in  $x_{k+1}$ .

As a polynomial in  $x_{k+1}$ , the degree of l.h.s. of (3.11) is  $(k-1)m + d'$  and is in  $a_{d', (1^{d'})}^{(k-2, k-3, \dots, 0)} e_{(1^{d'})} Q_T^{(k-2, k-3, \dots, 0); m}$ . Hence we have  $a_{d', (1^{d'})}^{(k-2, k-3, \dots, 0)} = 0$ .

Thus using the following lemma, we complete the proof of Proposition.3.9.

**Lemma 3.10** *Let  $k$  be an integer such that  $k \geq 3$ . We assume that for each integer  $l$  such that  $2 \leq l \leq n-1$  and each tableau  $\eta(n-1, l)$ , the statements of Proposition.3.9 holds.*

*Let  $r$  an integer such that  $1 \leq r \leq d' - 1$ . If we have the following equation:*

$$\sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_{s, n-2, k-1}} \sum_{i=0}^r \sum_{\substack{|\nu|=d'-s \\ l(\nu)=i}} a_{i,\nu}^\mu e_\nu Q_T^{\mu; m} = 0, \quad (3.12)$$

*then all constants  $a_{r,\nu}^\mu$  are equal to 0.*

Proof of lemma: We set

$$I = \sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_{s, n-2, k-1}} \sum_{i=0}^r \sum_{\substack{|\nu|=d'-s \\ l(\nu)=i}} a_{i,\nu}^\mu e_\nu Q_T^{\mu; m}.$$

From Proposition.3.3 (3), we have  $\deg_{x_{k+1}}(I) = (k-1)m + r$ . The leading coefficient of  $I$  in  $x_{k+1}$  is in

$$\sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_{s, n-2, k-1}} \sum_{\substack{|\nu|=d'-s \\ l(\nu)=r}} a_{r,\nu}^\mu e_\nu Q_T^{\mu; m}.$$

Recall that we have  $P_{s, n-2, k-1} = Q_{s, n-2, k-1} \cup P_{s, n-3, k-1}$  and this union is disjoint. Therefore we can rewrite this as

$$\begin{aligned} & \sum_{s=n-k}^{(k-1)(n-k)} \sum_{\mu \in Q_{s, n-2, k-1}} \sum_{\substack{|\nu^{(1)}|=d'-s \\ l(\nu^{(1)})=r}} a_{r, \nu^{(1)}}^\mu e_{\nu^{(1)}} Q_T^{\mu; m} \\ & + \sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu \in P_{s, n-3, k-1}} \sum_{\substack{|\nu^{(2)}|=d'-s \\ l(\nu^{(2)})=r}} a_{r, \nu^{(2)}}^\mu e_{\nu^{(2)}} Q_T^{\mu; m}. \end{aligned}$$

We set

$$\begin{aligned}
I_1 &= \sum_{s=n-k}^{(k-1)(n-k)} \sum_{\mu \in Q_{s,n-2,k-1}} \sum_{\substack{|\nu^{(1)}|=d'-s \\ l(\nu^{(1)})=r}} a_{r,\nu^{(1)}}^\mu e_{\nu^{(1)}} Q_T^{\mu;m} \\
I_2 &= \sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu \in P_{s,n-3,k-1}} \sum_{\substack{|\nu^{(2)}|=d'-s \\ l(\nu^{(2)})=r}} a_{r,\nu^{(2)}}^\mu e_{\nu^{(2)}} Q_T^{\mu;m}.
\end{aligned}$$

First, we show that the constants  $a_{r,\nu}^\mu$  in  $I_1$  are equal to 0.

If  $r > d' - n + k$ , we have  $|\mu| < \frac{(k-1)(k-2)}{2} + n - k$ . On the other hand, if  $\mu \in Q_{s,n-2,k-1}$ , we have  $|\mu| \geq \frac{(k-1)(k-2)}{2} + n - k$ . Therefore if  $r > d' - n + k$ , the sum in  $I_1$  is empty. We only need to consider the case when  $r \leq d' - n + k$ .

We define the following notions.

Let  $X = \{s_1, s_2, \dots, s_n\}$  be the set of  $n$  positive integers. For a partition  $\nu = (\nu_1, \nu_2, \dots)$ , we define

$$\begin{aligned}
e_{X,i} &= \sum_{1 \leq l_1 < \dots < l_i \leq n} x_{s_{l_1}} \cdots x_{s_{l_i}} \\
e_{X,\nu} &= \prod_i e_{X,\nu_i} \\
e_{X,i}^{(s_j)} &= e_i(x_{s_1}, \dots, x_{s_{j-1}}, x_{s_{j+1}}, \dots, x_{s_n}) \\
e_{X,\nu}^{(s_j)} &= \prod_{s_i} e_{X,\nu_i}^{(s_j)}.
\end{aligned}$$

In particular, if  $X = \{1, 2, \dots, n\}$ , then we simply write  $e_{X,i}^{(j)}$  as  $e_i^{(j)}$  and  $e_{X,\nu}^{(j)}$  as  $e_\nu^{(j)}$ .

When  $r \leq d' - n + k$ , the terms of the highest degree of  $I$  in  $x_1$  are in  $I_1$ . For  $\mu \in Q_{s,n-2,k-1}$ , there exists  $\mu' = (\mu'_1, \dots, \mu'_{k-2}) \in P_{n-3,k-2}$  such that  $\mu = (n-2, \mu'_1, \dots, \mu'_{k-2})$ . In particular, we have  $\mu' \in P_{s+k-n,n-3,k-2}$ . The leading coefficient of  $I_1$  in  $x_1$  is

$$\sum_{s=n-k}^{(k-1)(n-k)} \sum_{\mu' \in P_{s+k-n,n-3,k-2}} \sum_{\substack{|\nu^{(1)}|=d'-s \\ l(\nu^{(1)})=r}} b_{\nu^{(1)}}^{\mu'} e_{\nu^{(1)}-(1^r)}^{(1)} Q_T^{\mu';m}$$

where we set  $b_{\mu', \nu^{(1)}} = \frac{(-1)^{(k-1)m+1} m!}{\prod_{s=0}^m (mn+n-1-s)} a_{r, \nu^{(1)}}^{(n-2, \mu'_1, \dots)}$ . We can rewrite this as

$$\sum_{s=0}^{(k-2)(n-k)} \sum_{\mu' \in P_{s, n-3, k-2}} \sum_{\substack{|\nu^{(1)}| = d' - s + k - n \\ l(\nu^{(1)}) = r}} b_{\nu^{(1)}}^{\mu'} e_{\nu^{(1)} - (1^r)}^{(1)} Q_{T^1}^{\mu'; m}.$$

Since  $e_{\nu^{(1)} - (1^r)}^{(1)} = e_{\text{mem}(T^1), \nu^{(1)} - (1^r)}$ , this is rewritten as

$$\sum_{s=0}^{(k-2)(n-k)} \sum_{\mu' \in P_{s, n-3, k-2}} \sum_{\substack{|\nu^{(1)}| = d' - s + k - n \\ l(\nu^{(1)}) = r}} b_{\nu^{(1)}}^{\mu'} e_{\text{mem}(T^1), \nu^{(1)} - (1^r)} Q_{T^1}^{\mu'; m}.$$

The shape of  $T^1$  is  $(n-k+1, 1^{k-2})$ . Thus  $T^1$  has  $n-1$  boxes. By assumption on  $n$ , all  $b_{\nu^{(1)}}^{\mu'}$  are equal to 0. Thus we have  $a_{r, \nu^{(1)}}^{(n-2, \mu'_1, \dots)} = 0$ , hence we get  $I_1 = 0$ .

We next consider  $I_2$ . The leading coefficient of  $I_2$  in  $x_{k+1}$  is

$$\sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu \in P_{s, n-3, k-1}} \sum_{\substack{|\nu^{(2)}| = d' - s \\ l(\nu^{(2)}) = r}} c_{\nu^{(2)}}^{\mu} e_{\nu^{(2)} - (1^r)}^{(k+1)} Q_{T^{k+1}}^{\mu; m} \quad (3.13)$$

where we set  $c_{\nu^{(2)}}^{\mu} = (-1)^{(k-2)m} a_{r, \nu^{(2)}}^{\mu}$ .

Since  $e_{\nu^{(2)} - (1^r)}^{(k+1)} = e_{\text{mem}(T^{k+1}), \nu^{(2)} - (1^r)}$ , we can rewrite (3.13) as

$$\sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu \in P_{s, n-3, k-1}} \sum_{\substack{|\nu^{(2)}| = d' - s \\ l(\nu^{(2)}) = r}} c_{\nu^{(2)}}^{\mu} e_{\text{mem}(T^{k+1}), \nu^{(2)} - (1^r)} Q_{T^{k+1}}^{\mu; m}.$$

The tableau  $T^{k+1}$  has  $n-1$  boxes. By assumption on  $n$ , all  $c_{\nu^{(2)}}^{\mu}$  are equal to 0. Thus, all  $a_{r, \nu}^{\mu}$  are equal to 0.

Thus lemma follows. Therefore proposition also follows.  $\square$

From Theorem.3.8 and Proposition.3.9, we obtain the following corollary.

**Corollary 3.11** *Let  $T \in ST(\eta(n, k))$ .  $\gamma_T(\mathbf{QI}_m)$  is free module over  $\Lambda_n$  and  $\{Q_T^{\mu; m}\}_{\mu \in P(n-2; k-1)}$  is a free basis.*

Proof In this proof, we simply write  $Q_T^{\mu;m}$  as  $Q^\mu$ . Using Proposition.3.9,  $\{Q^\mu\}$  are free over  $\Lambda_n$ .

Since  $H(\gamma_T(\mathbf{QI}_m^*); t) = \sum_{s=0}^{(k-1)(n-k)} t^{(k-1)nm + \frac{k(k-1)}{2}} \sum_{s=0}^{(k-1)(n-k)} p_{s,n-2,k-1} t^s$ , we have

$$\gamma_T(\mathbf{QI}_m) = \bigoplus_{d \geq (k-1)nm + \frac{k(k-1)}{2}} \gamma_T(\mathbf{QI}_m)[d].$$

Let  $d$  be a non-negative integer such that  $d \geq (k-1)nm + \frac{k(k-1)}{2}$ . We show that the subspace of  $\gamma_T(\mathbf{QI}_m)[d]$  is generated by  $\{Q^\mu\}$  over  $\Lambda_n$ . We show this by induction on  $d$ .

When  $d = (k-1)nm + \frac{k(k-1)}{2}$ , the coefficient of  $t^{(k-1)nm + \frac{k(k-1)}{2}}$  in  $H(\gamma_T(\mathbf{QI}_m^*); t)$  is equal to 1. Therefore,  $\gamma_T(\mathbf{QI}_m)[d]$  is a space spanned by  $Q^{(k-2,k-1,\dots,0)}$ . Thus the statement follows in the case  $d = (k-1)nm + \frac{k(k-1)}{2}$ .

When  $d \geq (k-1)nm + \frac{k(k-1)}{2} + 1$ , we assume that the statement holds with all numbers less than  $d$ . We denote by  $V$  the vector space over  $\mathbb{Q}$  spanned by  $\{Q^\mu\}_{\mu \in P(n-2;k-1)}$ .

Take  $f \in \gamma_T(\mathbf{QI}_m)[d]$ . Since Theorem.2.6, we can find  $g \in V[d]$  such that  $[f] = [g]$  in  $\gamma_T(\mathbf{QI}_m^*)$ . Thus, we have  $f - g \in \langle e_1, \dots, e_n \rangle$ . This is expressed as

$$f - g = \sum_{s \geq 1} A_s u_s$$

where  $A_s \in \Lambda_n^s$  and  $u_s \in \gamma_T(\mathbf{QI}_m)$ .

Since  $\gamma_T(\mathbf{QI}_m)$  is a graded space, we can decompose  $u_s = \sum_{i \geq 0} u_{s,i}$  where  $u_{s,i} \in \gamma_T(\mathbf{QI}_m)[i]$ . We have  $\deg(A_s u_{s,i}) = s + i$ . Thus, we have

$$f - g = \sum_{l \geq 0} \sum_{s+i=l} A_s u_{s,i}.$$

Since  $f - g \in \gamma_T(\mathbf{QI}_m)[d]$ , we get  $\sum_{l \neq d} \sum_{s+i=l} A_s u_{s,i} = 0$ . Therefore, we have

$$f - g = \sum_{s \geq 1} A_s u_{s,d-s}.$$

$A_s$  has degree at least 1, therefore  $u_{s,d-s}$  has the degree less than  $d$ . By induction assumption,  $u_{s,d-s}$  can be expressed as

$$u_{s,d-s} = \sum_l B_l v_l$$

where  $B_l \in \Lambda_n$  and  $v_l \in V$ . Thus, the statement follows.  $\square$

## 4 The operator $L_m$

The operator  $L_m$  is defined as

$$L_m = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - 2m \sum_{1 \leq i < j \leq n} \frac{1}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)$$

This operator is discussed in [4] and [5]. It is related the quasiinvariants. In [5] Feigin and Veselov showed that the operator  $L_m$  preserves  $\mathbf{QI}_m$ . We consider how  $L_m$  acts on our polynomial  $Q_T^{\mu;m}$ . In [2], for  $T(1, 2)$  Bandlow and Musiker showed that the following formulas for the action of  $L_m$ .

**Theorem 4.1 ([2])** *Let  $k, m$  be non-negative integers.*

*Then we have  $L_m(Q_{T(1,2)}^{k;m}) = k(k-1)Q_{T(1,2)}^{k-2;m}$  for  $k \geq 2$  and  $L_m(Q_{T(1,2)}^{k;m}) = 0$  for  $k = 0, 1$ .*

We extend this formulas. We set  $T = T(1, 2, \dots, k)$ . To write formulas simply, we define the following polynomials.

**Definition 4.2** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}$ .

We define a polynomial  $Q_T^{\alpha;m}$  as follows:

$$Q_T^{\alpha;m} = \begin{vmatrix} R_{T;1,2}^{\alpha_1;m} & R_{D;1,2}^{\alpha_2;m} & \cdots & R_{T;1,2}^{\alpha_{k-1};m} \\ R_{T;2,3}^{\alpha_1;m} & R_{T;2,3}^{\alpha_2;m} & \cdots & R_{T;2,3}^{\alpha_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\alpha_1;m} & R_{T;k-1,k}^{\alpha_2;m} & \cdots & R_{T;k-1,k}^{\alpha_{k-1};m} \end{vmatrix} \quad (4.1)$$

when  $\alpha_i \geq 0$   $i = 1, \dots, k-1$ . Otherwise we define  $Q_T^{\alpha;m} = 0$ .

**Remark 4.3** If  $\alpha$  is a partition,  $Q_T^{\alpha;m}$  is equal to a polynomial defined in Definition.4.1. If  $\alpha \in \mathbb{Z}_{\geq 0}^{k-1}$ ,  $Q_T^{\alpha;m}$  is equal to  $Q_T^{\mu;m}$  up to a sign where  $\mu$  is a partition sorted  $\alpha$ .

We obtain the following formulas for the action of  $L_m$ . To write the formula simply, for  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}$  we define

$$\alpha^{(i,j)} = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \dots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \dots, \alpha_n).$$



**Theorem 4.4** Let  $\alpha = (\alpha_1, \dots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}$  and take  $T \in ST(\eta(n, k))$ . Then we have

$$\begin{aligned} L_m(Q_T^{\alpha; m}) &= \sum_{i=1}^n \alpha_i(\alpha_i - 1)Q_T^{(\alpha_1, \dots, \alpha_i-2, \dots, \alpha_n); m} + 2m \sum_{1 \leq i < j \leq n} (-\alpha_i Q_T^{\alpha^{(i,j)}; m} \\ &+ \sum_{\substack{\alpha_i-2 \geq s > t \geq 0 \\ s+t=\alpha_i+\alpha_j-2}} (s-t)Q_T^{(\alpha_1, \dots, \alpha_i-1, s, \alpha_i+1, \dots, \alpha_j-1, t, \alpha_j+1, \dots, \alpha_n); m}). \end{aligned}$$

This follows from following lemma. We define a polynomial  $R_{T;1,2,3}^{s,t;m}$  as

$$R_{T;1,2,3}^{s,t;m} = \begin{vmatrix} R_{T;1,2}^{s;m} & R_{T;1,2}^{t;m} \\ R_{T;1,3}^{s;m} & R_{T;2,3}^{t;m} \end{vmatrix}.$$

**Lemma 4.5** (1) we have

$$L_m(fg) = L_m(f)g + fL_m(g) + 2 \sum_{i=1}^n \left( \frac{\partial}{\partial x_i} f \right) \left( \frac{\partial}{\partial x_i} g \right).$$

(2) Let  $k$  be a non-negative integer and  $m$  be a non-negative integer. Then we have

$$k \int_{x_i}^{x_j} t^{k-1} \prod_{s=1}^n (t - x_s)^m = -m \sum_{r=1}^n \int_{x_i}^{x_j} t^k (t - x_r)^{m-1} \prod_{s \neq r} (t - x_s)^m dt.$$

(3) Let  $k, l$  be non-negative integers such that  $k > l$ . Then we have

$$\sum_{i=1}^n \left( \frac{\partial}{\partial x_i} R_{T;1,2}^{k;m} \right) \left( \frac{\partial}{\partial x_i} R_{T;1,3}^{l;m} \right) - \left( \frac{\partial}{\partial x_i} R_{T;1,3}^{k;m} \right) \left( \frac{\partial}{\partial x_i} R_{T;1,2}^{l;m} \right) \quad (4.2)$$

$$= m \left( -l R_{T;1,2,3}^{k-1, l-1; m} + \sum_{\substack{k-2 \geq s > t \geq 0 \\ s+t=k+l-2}} (s-t) R_{T;1,2,3}^{s,t;m} \right). \quad (4.3)$$

Proof (1) It follows from Leibniz's rule.

(2) It follows from following identity:

$$\int_{x_i}^{x_j} \frac{\partial}{\partial t} t^k \prod_{s=1}^n (t - x_s)^m = 0.$$

(3) When  $m = 0$ , it follows from  $R_{T;1,2}^{k;m} = \frac{x_2^{k+1} - x_1^{k+1}}{k+1}$ . We consider the case  $m \geq 1$ .

We show this formula by induction on  $k-l$ . We define  $f(t, x) = \prod_{s=1}^n (t - x_s)^m$  and  $f_i(t, x) = (t - x_i)^{m-1} \prod_{s \neq i} (t - x_s)^m$ . When  $k-l = 1$ , l.h.s. of (4.3) is equal to

$$\begin{aligned} & m^2 \sum_{i=1}^n \int_{x_1}^{x_2} t^k f_i(t, x) dt \int_{x_1}^{x_3} u^{k-1} f_i(u, x) du \\ & - m^2 \sum_{i=1}^n \int_{x_1}^{x_3} t^k f_i(t, x) dt \int_{x_1}^{x_2} u^{k-1} f_i(u, x) du. \end{aligned}$$

So this is equal to

$$\begin{aligned} & m^2 \sum_{i=1}^n \int_{x_1}^{x_2} t^{k-1} \{(t - x_i) + x_i\} f_i(t, x) dt \int_{x_1}^{x_3} u^{k-1} f_i(u, x) du \\ & - m^2 \sum_{i=1}^n \int_{x_1}^{x_3} t^{k-1} \{(t - x_i) + x_i\} f_i(t, x) dt \int_{x_1}^{x_2} u^{k-1} f_i(u, x) du \\ & = m^2 \sum_{i=1}^n \int_{x_1}^{x_2} t^{k-1} f(t, x) dt \int_{x_1}^{x_3} u^{k-1} f_i(u, x) du \\ & - m^2 \sum_{i=1}^n \int_{x_1}^{x_3} t^{k-1} f(t, x) dt \int_{x_1}^{x_2} u^{k-1} f_i(u, x) du. \end{aligned}$$

Using (2), we have

$$l.h.s. \text{ of (4.3) } = -m(k-1)R_{T;1,2,3}^{k-1,k-2;m}.$$

We consider the case  $k-l = 2$ . Calculating it in the same way, we have

$$\begin{aligned} l.h.s. \text{ of (4.3) } &= -m(k-2)R_{T;1,2,3}^{k-1,k-3;m} \\ &+ m^2 \sum_{i=1}^n \int_{x_1}^{x_2} t^{k-1} f_i(t, x) dt \int_{x_1}^{x_3} x_i u^{k-2} f_i(u, x) du \\ &- m^2 \sum_{i=1}^n \int_{x_1}^{x_3} t^{k-1} f_i(t, x) dt \int_{x_1}^{x_2} x_i u^{k-2} f_i(u, x) du. \end{aligned}$$

From  $x_i = u - (u - x_i)$ , we get

$$\begin{aligned}
l.h.s. \text{ of (4.3)} &= -m(k-2)R_{T;1,2,3}^{k-1,k-3;m} \\
&\quad + m^2 \sum_{i=1}^n \int_{x_1}^{x_2} t^{k-1} f_i(t, x) dt \int_{x_1}^{x_3} \{u - (u - x_i)\} u^{k-2} f_i(u, x) du \\
&\quad - m^2 \sum_{i=1}^n \int_{x_1}^{x_3} t^{k-1} f_i(t, x) dt \int_{x_1}^{x_2} \{u - (u - x_i)\} u^{k-2} f_i(u, x) du.
\end{aligned}$$

It is equal to  $-m(k-2)R_{T;1,2,3}^{k-1,k-3;m}$ . Thus theorem holds when  $k-l=2$ .

When  $k-l \geq 3$ , we assume that the formula (4.3) holds with all numbers less than  $k-l$ . Calculating l.h.s. of (4.3) in the same way, we have

$$\begin{aligned}
&l.h.s. \text{ of (4.3)} \\
&= -mlR_{T;1,2,3}^{k-1,l-1;m} + m(k-1)R_{T;1,2,3}^{k-2,l;m} \\
&\quad + \sum_{i=1}^n \left( \frac{\partial}{\partial x_i} R_{T;1,2}^{k-1;m} \right) \left( \frac{\partial}{\partial x_i} R_{T;1,3}^{l+1;m} \right) - \left( \frac{\partial}{\partial x_i} R_{T;1,3}^{k-1;m} \right) \left( \frac{\partial}{\partial x_i} R_{T;1,2}^{l+1;m} \right)
\end{aligned}$$

So theorem follows by induction assumption.  $\square$

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